Comment on "Two-loop renormalization-group analysis of the Burgers–Kardar-Parisi-Zhang equation"

Tao Sun
Department of Physics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6
(Received 28 July 1994)

In a recent paper, Frey and Täuber [Phys. Rev. E **50**, 1024 (1994)] conclude from a two-loop renormalization-group analysis that, to order $O(\epsilon^2)$, there is no strong coupling fixed point for the Kardar-Parisi-Zhang equation for substrate dimension $d = 2$. This contradicts previous studies. We comment on some points in the paper and clear up some conceptual confusion regarding the field theory renormalization-group technique.

PACS number(s): 05.40.+j, 05.70.Ln, 64.60.Ht, 68.35.Fx

Recently two groups, Sun and Plischke [1], and Frey and Täuber [2], performed two-loop renormalization-group analyses for the Kardar-Parisi-Zhang (KPZ) equation which is generally believed to describe the dynamics of driven interface where the lateral growth effect is dominant. However, they arrived at different conclusions for the practically important case of substrate dimension $d = 2$. Sun and Plischke obtain a strong coupling fixed point governing a rough growth regime with the roughening exponent $\chi \simeq 0.16$ and dynamic exponent $z \simeq 1.8$, while Frey and Täuber claim that, to order $O(\epsilon^2)$, there is no finite strong coupling fixed point in this dimension. Therefore, it is important to find out where the difference arises from. The aim of this paper is to comment on some points of Ref. [2] and clear up some basic concepts of the field theory renormalization-group technique.

The Kardar-Parisi-Zhang equation in the frame moving with the average growth velocity of the interface can be written as [3, 4]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \tfrac{1}{2}\lambda\, (\nabla h)^2 + \eta(\mathbf{x}, t) , \tag{1}$$

where $h(\mathbf{x}, t)$ is the interface height variable at space-time point $(\mathbf{x}, t)$, λ and ν are constants, and the noise $\eta(\mathbf{x}, t)$ satisfies the Gaussian distribution $\langle \eta(\mathbf{x}, t)\rangle = 0$, and $\langle \eta(\mathbf{x}, t)\ \eta(\mathbf{x}', t')\rangle = 2D\ \delta^d(\mathbf{x} - \mathbf{x}')\ \delta(t - t')$. It has been shown [3–5] that Eq. (1) is invariant under the Galilean transformation and the fluctuation-dissipation theorem exists in $d = 1$. Equation (1) can be transformed to Fourier space as

$$\begin{aligned}(-i\omega + \nu k^2)h(\mathbf{k}, \omega) = \eta(\mathbf{k}, \omega) \\ -\frac{\lambda}{2} \int_{\Omega_q} [\mathbf{q} \cdot (\mathbf{k} - \mathbf{q})] h(\mathbf{q}, \Omega) \\ \times h(\mathbf{k} - \mathbf{q}, \omega - \Omega) ,\end{aligned} \tag{2}$$

where $\int_{\Omega_q} = \int d\Omega d^d q/(2\pi)^{d+1}$. Starting from Eq. (2), the renormalized response function, two-point correlation function, and vertex function can be perturbatively calculated, whereby the dynamic scaling properties of the KPZ equation can be obtained.

The main reason leading Frey and Täuber to obtain the free-field result is that the response function $G_{11}(\mathbf{k}, \omega)$ is not renormalized in their analysis. In the loop-expansion formalism, the inverse of the response function, denoted by $\Gamma_{11}(\mathbf{k}, \omega)$, may be generally expressed as

$$\begin{aligned}\Gamma_{11}(\mathbf{k}, \omega) = -i\omega\ [1 + M_1 + M_2 + \cdots] \\ + \nu k^2\ [1 + N_1 + N_2 + \cdots] ,\end{aligned} \tag{3}$$

where M_i and N_i, $i = 1, 2, \ldots$, are contributions from i-loop calculation. Generally, the M's and N's have poles in $\epsilon = 2 - d$ due to the fact that the critical dimension $d_c = 2$ for the system. The task of the renormalization group is to remove these singularities. For the KPZ model in $d = 2$, we do have finite M_1 and N_1, indicating that the one-loop calculation does not contribute to the scaling properties of the model in this dimension. However, our calculation [1] shows that the two-loop terms M_2 and N_2 do have poles in ϵ. Thus the free-field results are corrected by the two-loop analysis and a strong coupling fixed point is obtained. Nevertheless, Frey and Täuber claim [2] that M_2 and N_2 are also finite, and then they arrive at a free-field result. We find that they obtain a finite M_2 because they choose a zero k normalization point and they obtain a finite N_2 because they use a "partial ϵ expansion" scheme, which we feel cannot be accepted both physically and mathematically in the renormalization-group formalism.

In their calculation, Frey and Täuber choose the point $(\mathbf{k} = \mathbf{0},\ \omega = -i\nu\mu^2)$ as the normalization point (NP) [2], where $\mathbf{k}$ and ω are the external momentum and the external frequency, respectively. One of the important properties of the KPZ equation is that the external momentum $\mathbf{k}$ appears as a factor in all integrands of the loop expansion expressions. Thus at their NP, i.e., $\mathbf{k} = \mathbf{0}$, all contributions beyond the tree approximation are gone and then [6]

$$\Gamma_{11}(\mathbf{k} = \mathbf{0}, \omega) = \Gamma_{11}^{(0)}(\mathbf{k} = \mathbf{0}, \omega) = -i\omega , \tag{4}$$

from which Frey and Täuber conclude that M_2 is finite. Such a choice of NP violates the principles of the renormalization-group theory. It is well known [8, 9] that there are two ways to choose NP depending on whether the system is massive or massless. For massive systems, the NP can be chosen at $\mathbf{k} = \mathbf{0}$ for simplicity. For massless systems, the NP must be chosen at finite $\mathbf{k}$ in order to avoid infrared problems. The KPZ model is massless, so that the NP must be chosen at finite external momentum $k^2 = \kappa^2$, as has been done in Ref [1].

Frey and Täuber have argued [7] that for the massless KPZ model the NP can be chosen at either finite $\mathbf{k}$ or finite ω; they choose the latter because it simplifies the calculation and the finite ω also helps to avoid infared problems. We do not agree with this argument. First of all, the external momentum $\mathbf{k}$ is a static variable appearing in both statics and dynamics, while the external frequency ω is a pure dynamic variable appearing only in dynamics. In other words, the two variables $\mathbf{k}$ and ω are not in the same position in the scaling analysis and the basic scaling variable is $\mathbf{k}$ rather than ω. In the static renormalization-group theory one chooses finite $\mathbf{k}$ as the NP for the massless system; in the dynamic renormalization-group analysis one should do the same thing for consistency. Second of all, for Frey and Täuber's argument to be acceptable the two ways of choosing the NP should be equivalent. That is, only when two kinds of choices of NP give the same result can one choose a simpler one. However, our calculation shows that the two kinds of NP lead to different results. Therefore, there is no choice between finite $\mathbf{k}$ and finite ω and one must choose finite $\mathbf{k}$ as the normalization point of the massless KPZ model.

We now discuss the "partial ϵ expansion" scheme which results in the conclusion of finite N_2 in Ref [2]. In the loop expansion of the vertex function $\Gamma_{11}(\mathbf{k},\omega)$, the d-dimensional integrals can be typically expressed in the following way:

$$I_{11}(\mathbf{k},\omega=0) = (2-d)\left[c_1\,\frac{1}{(2-d)} + c_2\,\frac{1}{(2-d)^2} + \cdots\right]\nu\,k^2\,, \qquad (5)$$

where c_1 and c_2 are constants analytic in $(2-d)$. The appearance of the factor $(d-2)$ outside of the square brackets is due to the form of vertex $[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]$ in Eq. (2). Frey and Täuber argue [2] that the $(d-2)$ factors in the square bracket are $\epsilon = 2-d$, while the one outside the bracket is not because it arises from the vertex $[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]$. Therefore they arrive at

$$I_{11}(\mathbf{k},\omega=0) = (2-d)\left[c_1\,\frac{1}{\epsilon} + c_2\,\frac{1}{\epsilon^2} + \cdots\right]\nu\,k^2\,. \qquad (6)$$

They describe this treatment as "clearly distinguishing between d and ϵ." Furthermore they conclude that in the case of $d = 2$ all contributions from the loop expansion disappear because of the factor $(2-d)$ in Eq. (6), no matter what kind of singularity exists in the square bracket. Consequently, they obtained finite N_2. This is their "partial ϵ expansion" scheme.

We feel that this partial ϵ expansion scheme is difficult to accept. Mathematically, it violates the basic principle of consistency. To calculate a d-dimensional integral, one cannot regard the factor $(2-d)$ as an ϵ somewhere and regard it as not an ϵ somewhere else. Furthermore, the factor $(2-d)$ outside of the square bracket arises from the vertex $[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]$, which originates from the gradient operator in the nonlinear term $(\nabla h)^2$ in the KPZ equation. It is well known that this nonlinear term describes the lateral growth effect, which is the essential physics of the KPZ universality class of driven interface system. It must be relevant to the large-scale, late-time scaling behavior of the KPZ model. Therefore, physically one also cannot simply regard it as an "artificial" dimensional factor and ignore the important role it plays in the KPZ model. The essence of the partial ϵ expansion is to remove the nonlinear term from the KPZ equation in $d = 2$ because of its gradient feature. That is, Frey and Täuber essentially discuss the Edwards-Wilkinson equation [11] in the case of $d = 2$, rather than the KPZ equation. Consequently, they obtain a free-field result for this dimension.

There are other points in the paper that we would like to discuss. First, the scaling analysis of the Callan-Symanzik equation is problematical. Let us now examine Eqs. (2.15)–(2.17) in Ref. [2]. To explain our point clearly, we rewrite their Eq. (2.15) as follows:

$$[h] = \Lambda^{-d/2-3}\,\nu_0^{-3/2}\,D_0^{1/2}\,. \qquad (7)$$

As Eq. (7) stands, to measure the canonical dimension of h, Frey and Täuber used three units, Λ, ν_0, and D_0. It is well known that [8–10] in the renormalization-group theory there is only one basic unit for measuring the canonical dimension of quantities, i.e., the momentum $\mathbf{k}$, or equivalently the cutoff Λ. Therefore, we feel that Eq. (7) should be corrected to be

$$[h] = \Lambda^{-d/2-3-3d_{\nu_0}/2+d_{D_0}/2}\,, \qquad (8)$$

where d_{ν_0} and d_{D_0} are canonical dimensions of the parameters ν_0 and D_0 measured in the basic unit of Λ. The other two equations should be corrected in the same way. Of course, the quantities d_{ν_0} and d_{D_0} cannot be fixed by the theory. But as has been pointed out in Ref. [1], the final dynamic scaling properties of the KPZ model do not depend on their exact values. However, they are relevant for the derivation of the dynamic scaling form.

Consequently, we believe that Frey and Täuber obtained incorrect expressions for the solutions of the Callan-Symanzik equation, Eqs. (3.54), (3.55), and (3.56) in Ref. [2]. As has been clearly explained by Amit [8] and Zinn-Justin [9], to solve the Callan-Symanzik equation, one needs the canonical dimension of the vertex function $\Gamma_{\tilde{h}h}$, which should be measured in only one basic unit of $\mathbf{k}$. However, since Frey and Täuber use three basic units to measure the canonical dimension of $\Gamma_{\tilde{h}h}$, some extra factors such as $\nu(l)$ and $D(l)$ appear in Eqs. (3.54) and (3.55) of Ref. [2]. Consequently, at a fixed point, an extra scaling factor $q^{-2\zeta_\nu^*+\zeta_D^*}$ is generated in their Eq. (3.56). These extra factors would make the final result inconsistent with Galilean invariance. Frey and Täuber do not realize this point because they obtained a free-field result.

Finally, we feel that the result for $d > 2$ in Ref. [2]

is doubtful. According to the standard field theoretic renormalization group [8, 9], the KPZ theory is *super-renormalizable* for $d < 2$, *renormalizable* in $d = d_c = 2$, and finally *nonrenormalizable* for $d > 2$. In other words, in principle the primitive divergences appearing in the loop expansion cannot be removed by a finite number of renormalization constants for $d > d_c$. That is, generally, the $d_c + \epsilon$ scheme in the field theoretic renormalization group does not work for the case of $d > d_c$. One famous example that a "nonrenormalizable theory" does get renormalized is the nonlinear σ model. Now it is understood that this is due to the underlying $O(N)$ symmetry of the model. In fact, the renormalizability of the nonlinear σ model above its critical dimension is confirmed by symmetry analysis, which is independent of the perturbation expansion [8, 9]. Regarding the KPZ model, whether it is practically renormalizable for $d > 2$ is still an open issue. Based on the work of Doty and Kosterlitz [12], it seems that the KPZ model is not renormalizable for $d > 2$. Therefore, the validity of the $2 + \epsilon$ expansion in Ref. [2] is questionable.

We hope this article can help to clear up some conceptual confusion regarding the useful field theory renormalization-group technique.

I thank Professor Michael Plischke for carefully reading the manuscript and making helpful suggestions. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

---

[1] T. Sun and M. Plischke, Phys. Rev. E **49**, 5046 (1994).

[2] E. Frey and U. C. Täuber, Phys. Rev. E. **50**, 1024 (1994).

[3] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[4] E. Medina, T. Hwa, M. Kardar, and Y. C. Zhang, Phys. Rev. A **39**, 3053 (1989).

[5] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).

[6] This is Eq. (2.27) in Ref. [2]. Frey and Täuber obtained this equation in the context of discussing the Galilean invariance and the Ward-Takahashi identity. They imply that this equation is a direct result of the intrinsic symmetries of the KPZ equation. However, careful examination reveals that this is not the case. From the Galilean invariance (Eqs. (2.18)–(2.19) in Ref. [2]) and the Ward-Takahashi indentity (Eq. (2.22) in Ref. [2]), one can obtain Eqs. (2.25) and (2.26), but not Eq. (2.27) of Ref. [2]. To obtain Eq. (2.27), one has to impose another condition, i.e., the external momentum $\mathbf{k} = \mathbf{0}$. That is, this equation is not a direct result of the Galilean invariance and the Ward-Takahashi identity. Instead, it is a result of zero external momentum, i.e., the choice of NP in Ref. [2].

[7] U. C. Täuber (private communication).

[8] D. J. Amit, *Field Theory, The Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1978).

[9] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1993).

[10] K. S. Ma, *Modern Theory of Critical Phenomena* (Addison-Wesley, Reading, MA, 1976).

[11] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London Ser. A **381**, 17 (1982).

[12] C. A. Doty and J. M. Kosterlitz, Phys. Rev. Lett. **69**, 1979 (1992).